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CLASSICAL ANALYSES OF LAMINATED BIMODULUS COMPOSITE-MATERIAL PL--ETC(U)
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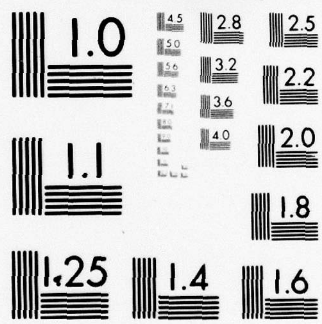

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6 CLASSICAL ANALYSES OF LAMINATED BIMODULUS COMPOSITE-MATERIAL PLATES

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by

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CLASSICAL ANALYSES OF LAMINATED BIMODULUS COMPOSITE-MATERIAL PLATES

Charles W. Bert

SUMMARY

A differential-equation formulation is presented for the equations governing the small-deflection elastic behavior of thin plates laminated of anisotropic bimodulus materials (which have different elastic stiffnesses depending upon the sign of the fiber-direction strains). As a basis for comparative evaluation of a finite-element formulation presented in Technical Report No. 3 of the contract, exact closed-form solutions are presented for two cross-ply-laminated plate problems: (1) a freely supported rectangular plate subjected to a sinusoidally distributed normal pressure, and (2) a fully clamped elliptic plate subjected to uniform normal pressure. For the special case of isotropic bimodulus material, simplified approximate solutions are deduced from the exact ones. Good agreement is obtained among the two solutions presented here, as well as with numerical results existing in the literature for special cases and with the finite-element results. Also, for the first time is presented a closed-form solution for an elliptic plate arbitrarily laminated of anisotropic ordinary (not bimodulus) material.

1. INTRODUCTION

As the use of fiber-reinforced composite materials in structures becomes more widespread, the prediction of the behavior of plates constructed of such materials becomes increasingly important. One of the characteristics of certain composite materials, known as bimodulus materials, is that they exhibit quite different elastic properties when loaded along the fiber direction in tension as opposed to compression [1-4]. A plate subjected to a loading which produces plate bending obviously experiences both tension and compression; therefore, a more accurate analysis should take this into consideration.

A macroscopic material model appropriate for bimodulus fiber-reinforced composites has been introduced by the present author and found to agree well with experimental results [5]. The purpose of the current project is to develop finite-element analyses for plates laminated of composite materials described by the model of [5]. In order to provide a sound basis of evaluation of the finite-element analyses developed [6], it is desirable to compare the finite-element numerical results with those obtained by other means, preferably by closed-form solution [7]. The purpose of the present report is to present some exact closed-form solutions, as well as some simplified approximate ones, and to compare the numerical results with those obtained by other means.

Unfortunately, the existing literature available in English on bending of bimodulus plates is quite sparse and, with one exception, is limited to bimodulus isotropic material [8-12]. Shapiro [8] considered the very simple problem of a circular plate subjected to a pure radial

bending moment at its edge, but he used Love's stress-function formulation rather than plate theory. Kamiya [9] treated large deflections (geometric nonlinearity) of uniformly loaded, clamped-edge circular plates, using an iterative finite-difference technique. In [10], Kamiya applied the energy method to large deflections of simply supported rectangular plates subjected to sinusoidally distributed loading. In [11], Kamiya included the effect of thickness shear deformation, but only for the simple case of cylindrical bending. The only analysis applicable to anisotropic bimodulus material is the work of Jones and Morgan [12], who treated cylindrical bending of a thin, cross-ply laminate. Apparently, the present work and its finite-element companion work [6] are the first to consider anisotropic bimodulus plates finite in both directions.

In the realm of plates laminated of ordinary* anisotropic materials, the theory due to Reissner and Stavsky [13] is generally recognized as the classical, linear (small-deflection) thin-plate theory. Although there have been numerous approximate solutions of this theory, only a relatively few closed-form solutions have appeared. Notable among these are the works of Whitney [14] and Whitney and Leissa [15] for both antisymmetric cross-ply and antisymmetric angle-ply rectangular plates with certain (different) kinds of simply supported edges, and Kicher's work [16] on antisymmetric cross-ply elliptic plates with clamped edges. For an infinitely long strip of finite width, Padovan [17] presented a solution for the case of an arbitrary laminate, i.e. one having a complete array of

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* Here an <u>ordinary</u> material is understood to mean one with the same elastic properties in tension and compression.

stretching, bending-stretching coupling, and bending stiffnesses. In Appendix C of the present report, a closed-form solution is presented for clamped-edge elliptic plates arbitrarily laminated of ordinary anisotropic material.

2. FORMULATION

2.1 Laminate Behavior

First, let us consider the case of ordinary (not bimodulus) material. A single-layer plate constructed of an ordinary material that is macroscopically homogeneous is obviously symmetric about the midplane of the plate and thus there is no coupling between bending and stretching. Likewise a plate consisting of multiple layers of ordinary materials of various thicknesses arranged symmetrically about the midplane has no bending-stretching coupling. However, in the case of a general laminate, i.e. one not symmetric about the midplane, bending-stretching coupling is induced.

Now, let us consider the case of a single layer of bimodulus material. The different properties in tension and compression cause a shift in the neutral surface away from the geometric midplane and symmetry about the midplane no longer holds. The result of this is that a single-layer bimodulus-material plate exhibits bending-stretching coupling of the orthotropic type, i.e. analogous to a two-layer cross-ply plate (one layer at 0° and the other at 90°) of ordinary orthotropic material.

Using the present investigator's fiber-governed symmetric-matrix macroscopic material model [5], it can be assumed that there are <u>two</u> symmetric plane-stress reduced stiffness matrices: one when the <u>fibers</u> are in tension along their length and another when they are in compression in the same direction. Invoking the Voigt hypothesis in

the fiber direction, for which it is well established, it is assumed that the fiber-direction normal strains in the fibers and in the matrix are identical. Then the criterion for changing from tension to compression can be taken to be the fiber-direction normal strain in each layer. This is a much more convenient criterion to apply than is the fiber-direction normal stress in the fiber.

Now the stress-strain relation for a thin orthotropic bimodulus material may be written as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11k\ell} & Q_{12k\ell} & 0 \\ Q_{12k\ell} & Q_{22k\ell} & 0 \\ 0 & 0 & Q_{66k\ell} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \tag{2.1}$$

The nomenclature is listed in Appendix A. The contracted notation of classical composite-material plate theory is used here, with the addition of subscript k (=t or c depending upon whether the fiber-direction strain is tensile or compressive), and ℓ denotes the layer number (=1 to N, where N is the total number of layers). From Eq. (2.1), it can be seen that there are now eight independent elastic stiffness coefficients for each orthotropic layer, a set of four for tension and another set of four for compression. This is in contrast to an ordinary orthotropic material which has a total of only four independent coefficients per layer.

The stretching, bending-stretching coupling, and bending stiffnesses of the laminate are defined formally in exactly the same way as in the case of a laminate of ordinary material

$$A_{ij} \equiv \int_{-h/2}^{h/2} Q_{ijk\ell}\, dz \;;\quad B_{ij} \equiv \int_{-h/2}^{h/2} zQ_{ijk\ell}\, dz$$

$$D_{ij} \equiv \int_{-h/2}^{h/2} z^2 Q_{ijk\ell}\, dz \tag{2.2}$$

Now, in addition to performing the integrations in a piecewise manner from layer to layer, one also has to take into account the possibility of different properties (tension or compression) within a layer. This is carried out in detail for a two-layer cross-ply laminate in Appendix B. It is to be emphasized that in a bimodulus-material laminate, there are no additional sets of A_{ij}, B_{ij}, and D_{ij} as compared to an ordinary-material laminate, since these stiffnesses are the results of thickness-direction integrations. However, as mentioned before with respect to the B_{ij}, there may be more non-zero stiffnesses in the bimodulus case.

The laminate constitutive relations for an arbitrary laminate of anisotropic material are

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{Bmatrix} \tag{2.3}$$

It is noted that the stiffness matrix is symmetric.

2.2 Thin-Plate Behavior

The thin-plate equilibrium equations, in the absence of body forces and body moments, are

$$N_{1,x} + N_{6,y} = 0 \quad ; \quad N_{6,x} + N_{2,y} = 0$$

$$M_{1,xx} + 2M_{6,xy} + M_{2,yy} = q \tag{2.4}$$

Here $(\ )_{,xy}$ denotes $\partial^2(\ )/\partial x \partial y$.

For a thin plate (Kirchhoff hypothesis) undergoing small deflections,

$$\varepsilon_1^0 = u_{,x} \quad ; \quad \varepsilon_2^0 = v_{,y} \quad ; \quad \varepsilon_6^0 = u_{,y} + v_{,x}$$

$$\kappa_1 = -w_{,xx} \quad ; \quad \kappa_2 = -w_{,yy} \quad ; \quad \kappa_6 = -2w_{,xy} \tag{2.5}$$

Substituting Eqs. (2.3) and (2.5) into Eqs. (2.4), one obtains the following governing partial differential equations in terms of the midplane displacements u, v, w:

$$[L_{rs}] \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q \end{Bmatrix} \qquad (r,s=1,2,3) \tag{2.6}$$

Here $[L_{rs}]$ is a symmetric linear differential operator matrix with components as follows:

$$L_{11} \equiv A_{11}d_x^2 + 2A_{16}d_x d_y + A_{66}d_y^2$$

$$L_{12} \equiv A_{16}d_x^2 + (A_{12} + A_{66})\, d_x d_y + A_{26}d_y^2$$

$$L_{13} \equiv -B_{11}d_x^3 - 3B_{16}d_x^2 d_y - (B_{12} + 2B_{66})\, d_x d_y^2 - B_{26}d_y^3$$

$$L_{22} \equiv A_{66}d_x^2 + 2A_{26}d_x d_y + A_{22}d_y^2$$

$$L_{23} \equiv - B_{16}d_x^3 - (B_{12} + 2B_{66})\, d_x^2 d_y - 3B_{26}d_x d_y^2 - B_{22}d_y^3$$

$$L_{33} \equiv D_{11}d_x^4 + 4D_{16}d_x^3 d_y + 2(D_{12} + 2D_{66})\, d_x^2 d_y^2 + 4D_{26}d_x d_y^3$$

$$+ D_{22}d_y^4 \tag{2.7}$$

Here $d_x^2 \equiv \partial^2(\ \ )/\partial x^2$, etc.

The solution of Eqs. (2.6) subject to appropriate boundary conditions constitute the correct solution of the problem. Due to the presence of bending-stretching coupling (at least one $B_{ij} \neq 0$), it is necessary to specify four boundary conditions for all points on the boundary: two flexural and two in-plane.

2.3 Criterion for Homogeneity in the Plane

In deriving the preceding equations, it was tacitly assumed that the plate stiffnesses (A_{ij}, B_{ij}, D_{ij}) are all independent of the x, y location in the plane. However, it is also clear from the role of the fiber-direction neutral-surface position z_{nf} (see Appendix B) that these stiffnesses depend upon z_{nf}. Hence, to ensure that the plate stiffnesses are in fact independent of x and y, it is necessary that z_{nf} also be independent of x and y. The condition for this requirement to be met is derived here.

The Kirchhoff hypothesis, applied to the fiber direction, is

$$\varepsilon_f = \varepsilon_f^0 + z\kappa_f \tag{2.8}$$

Thus, to determine the z-position of the fiber-direction neutral surface, one sets

$$0 = \varepsilon_f^0 + z_{nf}\kappa_f$$

or

$$z_{nf} = - \epsilon_f^0/\kappa_f \tag{2.9}$$

Now it is clear that in order for z_{nf} to be independent of x and y, ϵ_f^0/κ_f must also be independent of x and y. Although this restriction on the present equations is rather severe, the present investigator has found two classes of problems for which this condition is satisfied and furthermore has obtained closed-form solutions for them. They are cross-ply laminated plates with the following conditions:

Plate Geometry	Loading	Boundary Conditions
Rectangular	Sinusoidally distributed	Freely supported
Elliptic	Uniformly distributed	Fully clamped

In ensuing sections, these solutions are presented, discussed, and compared with solutions obtained by other classical methods and by the mixed finite-element method.

3. RECTANGULAR PLATE WITH SINUSOIDALLY DISTRIBUTED LOADING

3.1 Exact Closed-Form Solution for Orthotropic Cross-Ply Bimodulus Laminate

The boundary conditions considered here are the same as those used by Whitney and Leissa [15] for a cross-ply plate, i.e. all edges freely supported on a rectangular planform of dimensions a and b in the x and y directions:

$$\begin{aligned} N_1(0,y) = N_1(a,y) = 0 \quad &; \quad v(0,y) = v(a,y) = 0 \\ u(x,0) = u(x,b) = 0 \quad &; \quad N_2(x,0) = N_2(x,b) = 0 \\ w(0,y) = w(a,y) &= w(x,0) = w(x,b) = 0 \\ M_1(0,y) = M_1(a,y) = 0 \quad &; \quad M_2(x,0) = M_2(x,b) = 0 \end{aligned} \tag{3.1}$$

The loading is considered to be sinusoidally distributed in both the x and y directions as follows:

$$q = q_{mn} \sin \alpha x \sin \beta y \qquad (3.2)$$

Here

$$\alpha \equiv m\pi/a \quad , \quad \beta \equiv n\pi/b \qquad (3.3)$$

and m and n are integers.

It is clear that any arbitrary loading distribution can be treated by merely decomposing it into its Fourier series components, as was done by Navier in his famous solution of a simply supported, homogeneous, isotropic rectangular plate; see [18].

The laminate considered is a cross-ply one, i.e. all layers have the fibers oriented at either 0° or 90° to the x axis. Thus, all stiffnesses with subscripts 16 and 26 vanish.

The following forms of displacement functions satisfy Eqs. (2.6) and (3.1) exactly and yield constant values of z_{nx} and z_{ny}:

$$\begin{aligned} u(x,y) &= U_{mn} \cos \alpha x \sin \beta y \\ v(x,y) &= V_{mn} \sin \alpha x \cos \beta y \\ w(x,y) &= W_{mn} \sin \alpha x \sin \beta y \end{aligned} \qquad (3.4)$$

Substituting Eqs. (3.4) into Eqs. (2.6), one obtains

$$[C_{rs}]\{U,V,W\}^T = \{0,0,q\}^T \qquad (r,s=1,2,3) \qquad (3.5)$$

Here subscripts mn have been omitted for brevity and $[C_{rs}]$ is a symmetric matrix with elements given by

$$C_{11} \equiv -A_{11}\alpha^2 - A_{66}\beta^2 \quad ; \quad C_{12} \equiv -(A_{12} + A_{66})\alpha\beta$$

$$C_{13} \equiv B_{11}\alpha^3 + (B_{12} + 2B_{66})\alpha\beta^2 \quad ; \quad C_{22} \equiv -A_{66}\alpha^2 - A_{22}\beta^2 \tag{3.6}$$

$$C_{23} \equiv (B_{12} + 2B_{66})\alpha^2\beta + B_{22}\beta^3 \quad ; \quad C_{33} \equiv -D_{11}\alpha^4 - 2(D_{12} + 2D_{66})\alpha^2\beta^2 - D_{22}\beta^4$$

From matrix Eq. (3.5), the following ratios of displacements can be established:

$$\begin{aligned} U/W &= (C_{12}C_{23} - C_{13}C_{22})/(C_{11}C_{22} - C_{12}^2) \\ V/W &= (C_{12}C_{13} - C_{11}C_{23})/(C_{11}C_{22} - C_{12}^2) \end{aligned} \tag{3.7}$$

However, from Kirchhoff's hypothesis

$$\varepsilon_1 = u,_x - zw,_{xx} \quad ; \quad \varepsilon_2 = v,_y - zw,_{yy} \tag{3.8}$$

Thus, the neutral-surface positions, for the longitudinal (x) and transverse (y) directions respectively, are

$$\begin{aligned} z_{nx} &= u,_x/w,_{xx} = U/W\alpha \\ z_{ny} &= v,_y/w,_{yy} = V/W\beta \end{aligned} \tag{3.9}$$

Eliminating U/W and V/W from Eqs. (3.7) and (3.9), one obtains

$$\begin{aligned} (C_{11}C_{22} - C_{12}^2)\alpha z_{nx} &= C_{12}C_{23} - C_{13}C_{22} \\ (C_{11}C_{22} - C_{12}^2)\beta z_{ny} &= C_{12}C_{13} - C_{11}C_{23} \end{aligned} \tag{3.10}$$

Now it is convenient to nondimensionalize z_{nx} and z_{ny} as follows

$$Z_x \equiv z_{nx}/h \quad , \quad Z_y \equiv z_{ny}/h \tag{3.11}$$

Thus, Eqs. (3.10) become

$$(C_{11}C_{22} - C_{12}^2)h\alpha Z_x = C_{11}C_{23} - C_{13}C_{22}$$
$$(C_{11}C_{22} - C_{12}^2)h\beta Z_y = C_{12}C_{13} - C_{11}C_{23} \quad (3.12)$$

In Appendix B, it is shown that the A_{ij} have constant terms and terms linear in Z_x and Z_y, the B_{ij} have constant terms and terms quadratic in Z_x and Z_y, and the D_{ij} have constant terms and terms cubic in Z_x and Z_y. Substituting these equations and Eqs. (3.6) into Eqs. (3.12), one obtains two simultaneous, nonlinear algebraic equations in Z_x and Z_y. The solutions of these two equations represent the positions of the two neutral surfaces for the planform, boundary conditions, laminate, and loading concerned. Due to the lengthy algebraic forms, it is more convenient computationally to use an iterative procedure to determine Z_x and Z_y. Once this set of values is found, all of the other quantities can be found if desired.

3.2 Simplified Approximate Solution for Isotropic Bimodulus Material

The only bimodulus rectangular plate solution available in the literature is Kamiya's approximate solution [10] for the isotropic square plate. Also, in view of the complicated nature of the exact solution given in Section 3.1, it seemed worthwhile to derive a simplified approximate version for the isotropic rectangular plate.

The basis for the simplification was the assumption that Poisson's ratio could be factored out of the plate stiffness expressions. Thus,

$$A_{11} = A_{22} = A \quad , \quad A_{12} \simeq \nu A \quad , \quad A_{66} \simeq (1-\nu)(A/2)$$
$$B_{11} = B_{22} = B \quad , \quad B_{12} \simeq \nu B \quad , \quad B_{66} \simeq (1-\nu)(B/2) \quad (3.13)$$
$$D_{11} = D_{22} = D \quad , \quad D_{12} \simeq \nu D \quad , \quad D_{66} \simeq (1-\nu)(D/2)$$

Using approximations (3.13) in Eqs. (3.6) and thence in Eqs. (3.7) and (3.8), one finds

$$U/W = B\alpha/A = \alpha z_{nx} \quad ; \quad V/W = B\beta/A = \beta z_{ny}$$

Thus,

$$z_{nx} = z_{ny} = B/A = z_n \tag{3.14}$$

It is interesting to note that in this case not only is z_{nx} equal to z_{ny} but also its value depends only upon B/A (and thus upon the lamination scheme and material properties) and not upon any plate geometric dimensions. Furthermore, it can be shown that the solution is identical to that of an ordinary isotropic plate except that the actual flexural rigidity (D) is replaced by the "reduced flexural rigidity" $D^{(R)}$ defined as follows:

$$D^{(R)} = D - B^2/A \tag{3.15}$$

In view of Eq. (3.14), Eq. (3.15) can be written in this more convenient form

$$D^{(R)} = D - BhZ \tag{3.16}$$

where $Z \equiv z_n/h$.

For the present case, the equations derived in Appendix B simplify as follows:

$$\begin{gathered} A/h = \bar{Q} + \Delta QZ \quad ; \quad B/h^2 = -(1/8)\,\Delta Q\,(1 - 4Z^2) \\ 12D/h^3 = \bar{Q} + 4\Delta QZ^3 \end{gathered} \tag{3.17}$$

Here

$$\bar{Q} \equiv (1/2)(Q_c + Q_t) \quad , \quad \Delta Q \equiv Q_c - Q_t \tag{3.18}$$

Eqs. (3.14) and (3.17) can be combined to yield the following quadratic equation for Z:

$$Z = -(\bar{Q}/\Delta Q) \pm [(\bar{Q}/\Delta Q)^2 - (1/4)]^{\frac{1}{2}} \tag{3.19}$$

Also Eq. (3.16) can be rewritten as

$$12D^{(R)}/h^3 = \bar{Q} + \Delta Q[(3/2) - 2Z^2]Z \tag{3.20}$$

3.3 Numerical Results

Since Kamiya's energy-method results [10] are given for two different dimensionless loadings, $P = (q/E_c)(a/h)^4 = 16.91$ and 33.82, they can be used to obtain equivalent small-deflection results in conjunction with the following general approximate expression given by Timoshenko and Woinowsky-Krieger [19]

$$W/h = KP/[1 + C(W/h)^2] \tag{3.21}$$

Here the two constants can be found from two sets of values of P and W/h. For example, for $E_t/E_c = 2$, $W/h = 0.275$ for $P = 16.91$ and $W/h = 0.440$ for $P = 33.82$. Using these values in Eq. (3.21) twice leads to two simultaneous linear algebraic equations for K and C. In the present example

$$(0.275)^3C - 16.91K = -0.275$$

$$(0.440)^3C - 33.82K = -0.440$$

Thus, $C = 2.52$ and $K = 0.0194$. It is noted that K is merely the dimensionless linear deflection

$$K = \frac{W/h}{(q/E_c)(a/h)^4} = \frac{E_cWh^3}{qa^4}$$

For $E_t/E_c = 0.5$, $W/h = 0.54$ and 0.81 for $P = 16.91$ and 33.82, respectively. Thus, $K = 0.0435$.

Table 3.1 presents a comparison of numerical results for isotropic bimodulus square plates subjected to sinusoidal loading (m=n=1), as obtained by four different methods.

Table 3.1 Dimensionless Maximum Deflections, $E_c Wh^3/qa^4$, for Sinusoidally Loaded, Simply Supported Square Plates of Isotropic Bimodulus Materials Having $\nu_c = 0.20$

E_t/E_c	(1)	(2)	(3)	(4)
0.5	0.0435	0.0439	0.0439	0.0435
1.0	-	0.0296	0.0296	0.0293
2.0	0.0194	0.0204	0.0204	0.0202
2.5	-	0.0180	0.0180	0.0178

Method:

(1) Deduced from Kamiya's large-deflection energy-solution results [10]*.

(2) Exact closed-form solution, using G_c and G_t from Eqs. (4.16).

(3) Simplified approximate solution.

(4) Mixed finite-element solution (for a/h = 100).

The relatively close agreement among the results obtained by all four methods is encouraging.

The question arises as to the value of $Z(=z_n/h)$ associated with

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* It should be noted that Kamiya's Eq. (20) was subscripted improperly, but was recently confirmed to be only a typographical error [20].

the above deflections. Since Kamiya did not report Z, one can compare only the results from the last three methods; see Table 3.2.

Table 3.2 Values of the Dimensionless Neutral-Surface Position for the Plates of Table 3.1

E_t/E_c	(2)	(3)	(4)
0.5	- 0.090	- 0.090	- 0.090
1.0	0	0	0
2.0	0.102	0.102	0.102
2.5	0.141	0.141	0.141

It is seen in Table 3.2 that there is perfect agreement among all three methods.

As examples of actual bimodulus composite materials, the elastic-property data presented in Table 3.3 are used. They were obtained from the experimental results of Patel et al. [2], using the approach developed for Model 2 in [5].

Table 3.3 Elastic Properties of Two Fiber-Reinforced Bimodulus Composite Materials

Property	Aramid-Rubber	Polyester-Rubber
Tension Properties:		
Major Young's Modulus (GPa)	3.58	0.617
Minor Young's Modulus (GPa)	0.00909	0.00800
Major Poisson's Ratio (dimensionless)	0.416	0.475
Shear Modulus (GPa)	0.00370	0.00262
Compression Properties:		
Major Young's Modulus (GPa)	0.0120	0.0369
Minor Young's Modulus (GPa)	0.0120	0.0106
Major Poisson's Ratio (dimensionless)	0.205	0.185
Shear Modulus (GPa)	0.00370	0.00267

Numerical results, as calculated by closed-form and mixed finite-element methods, for single-layer rectangular plates constructed of these materials are presented in Table 3.4. As can be seen, there is excellent agreement between both Z_x and dimensionless deflection as predicted by the closed-form and finite-element methods. In fact, the largest differences appearing in the table are only 0.2% for Z_x and 98% for $WE_{22_c}h^3/qb^4$.

Results for the case of cross-ply laminated plates with the bottom layer (layer 1) at 0° (aligned with the x axis) and the top layer (layer 2) at 90° are presented in Table 3.5. It is seen that not only are the trends in Table 3.5 similar to those in Table 3.4, but also even the numerical values (especially Z_x) are very close. This may be important for preliminary design of laminated bimodulus plates: for these two typical bimodulus composite materials in this plate loading and configuration, the bimodulus effect is much more pronounced than the lamination effect. Thus, to a first approximation, one may design a multilayer cross-ply plate of bimodulus material as if it were a single-ply one.

Table 3.4. Fiber-Direction Neutral-Surface Positions and Deflections for Simply-Supported Rectangular Plate of Single-Layer 0° Aramid-Rubber and Polyester-Rubber, as Determined by Two Different Methods

Aspect Ratio c	Z_x Closed Form	Z_x Finite Element*	$WE_{22_c}h^3/qb^4$ Closed Form	$WE_{22_c}h^3/qb^4$ Finite Element*
		Aramid-Rubber:		
0.5	0.4457	0.4454	0.001881	0.001874
0.6	0.4457	0.4451	0.003661	0.003640
0.7	0.4457	0.4447	0.006253	0.006211
0.8	0.4444	0.4440	0.009679	0.009605
0.9	0.4444	0.4431	0.01387	0.01376
1.0	0.4424	0.4420	0.01870	0.01854
1.2	0.4398	0.4393	0.02956	0.02928
1.4	0.4368	0.4362	0.04089	0.04049
1.6	0.4334	0.4328	0.05170	0.05120
1.8	0.4298	0.4292	0.06143	0.06085
2.0	0.4260	0.4254	0.06995	0.06931
		Polyester-Rubber:		
0.5	0.3040	0.3041	0.000816	0.000821
0.6	0.3040	0.3041	0.001655	0.001656
0.7	0.3040	0.3039	0.002975	0.002968
0.8	0.3040	0.3036	0.004888	0.004866
0.9	0.3040	0.3031	0.007478	0.007434
1.0	0.3027	0.3026	0.01079	0.01071
1.2	0.3014	0.3012	0.01954	0.01935
1.4	0.2998	0.2995	0.03056	0.03021
1.6	0.2979	0.2976	0.04278	0.04224
1.8	0.2958	0.2954	0.05505	0.05431
2.0	0.2936	0.2931	0.06652	0.06560

* For b/h = 100

Table 3.5. Neutral-Surface Positions & Dimensionless Deflections for Simply-Supported Rectangular Plates of Cross-Ply Laminated Aramid-Rubber and Polyester-Rubber Determined by Two Methods

Aspect Ratio c	Z_x C.F.*	Z_x F.E.*	Z_y C.F.*	Z_y F.E.*	$WE_{22_c}h^3/qb^4$ C.F.*	$WE_{22_c}h^3/qb^4$ F.E.*
			Aramid-Rubber:			
0.5	0.4438	0.4390	- 0.07137	- 0.07202	0.001808	0.001801
0.6	0.4438	0.4419	- 0.06052	- 0.06208	0.003486	0.003464
0.7	0.4423	0.4419	- 0.05165	- 0.05302	0.005925	0.005880
0.8	0.4413	0.4413	- 0.04489	- 0.04608	0.009162	0.009082
0.9	0.4401	0.4404	- 0.03964	- 0.04145	0.01315	0.01303
1.0	0.4389	0.4392	- 0.03546	- 0.03712	0.01780	0.01761
1.2	0.4362	0.4360	- 0.02925	- 0.03060	0.02838	0.02807
1.4	0.4332	0.4334	- 0.02487	- 0.02592	0.03961	0.03917
1.6	0.4300	0.4302	- 0.02163	- 0.02295	0.05046	0.04990
1.8	0.4265	0.4266	- 0.01917	- 0.02029	0.06032	0.05967
2.0	0.4228	0.4229	- 0.01815	- 0.01818	0.06894	0.06826
			Polyester-Rubber:			
0.5	0.3650	0.3719	- 0.1412	- 0.1310	0.001902	0.001886
0.6	0.3650	0.3653	- 0.1244	- 0.1277	0.003672	0.003648
0.7	0.3638	0.3642	- 0.1139	- 0.1171	0.006227	0.006175
0.8	0.3638	0.3632	- 0.1060	- 0.1085	0.009542	0.009448
0.9	0.3622	0.3626	- 0.1003	- 0.1039	0.01348	0.01333
1.0	0.3622	0.3618	- 0.09605	- 0.09925	0.01783	0.01762
1.2	0.3594	0.3603	- 0.09029	- 0.09415	0.02680	0.02646
1.4	0.3573	0.3583	- 0.08670	- 0.08958	0.03497	0.03451
1.6	0.3550	0.3560	- 0.08432	- 0.08628	0.04167	0.04112
1.8	0.3525	0.3541	- 0.08268	- 0.08070	0.04690	0.04625
2.0	0.3498	0.3541	- 0.08150	- 0.07757	0.05090	0.05021

* C.F. denotes closed-form solution; F.E. signifies finite-element solution with b/h = 100.

4. ELLIPTIC PLATE WITH UNIFORMLY DISTRIBUTED LOADING

4.1 Exact Closed-Form Solution for Orthotropic Cross-Ply Bimodulus Laminate

The boundary conditions considered here are the same as the set used by Kicher [16], namely, fully clamped along the entire boundary

$$w = w_{,n} = u = v = 0 \quad \text{on} \quad (x/a)^2 + (y/b)^2 = 1 \tag{4.1}$$

where n denotes the outer normal at any boundary point and a and b are the plate semi-axis lengths in the x and y directions.

The loading considered is a uniformly distributed one with intensity q.

The laminate investigated here is a cross-ply one, as it is shown in Appendix C that this is the most general one for which the closed-form solution presented here is applicable to the bimodulus case.

The Bryan-Kicher solution, which can be obtained by setting $U_2 = V_2 = 0$ in Eqs. (C-1) of Appendix C, is given by

$$u = U_1[1 - (x/a)^2 - (y/b)^2](x/a) \quad ; \quad v = V_1[1 - (x/a)^2 - (y/b)^2](y/b)$$
$$w = W[1 - (x/a)^2 - (y/b)^2]^2 \tag{4.2}$$

This is the same form of displacement functions used by Kicher [16] in his closed-form analysis of an antisymmetric cross-ply laminated plate. The normal-deflection function had been originated by G. H. Bryan for isotropic plates according to Love [21] and was later used for orthotropic and anisotropic plates by Ohasi [22] and Lekhnitskii [23].

The applicable coefficients, C_{rs}, appearing in Eq. (3.5) are

given by those in Eqs. (C-3) with r,s = 1,2,3 and all terms with subscripts 16 and 26 set equal to zero. Now Eqs. (3.7) and (3.8) also hold for the present case and Eqs. (3.9) are replaced by

$$z_{nx} = u,_x/w,_{xx} = -(U_1/W)(a/4)$$
$$z_{ny} = v,_y/w,_{yy} = -(V_1/W)(b/4) \tag{4.3}$$

Equations (3.12) are replaced by

$$(C_{11}C_{22} - C_{12}^2)(4h/a)Z_x = -(C_{12}C_{23} - C_{13}C_{22})$$
$$(C_{11}C_{22} - C_{12}^2)(4h/b)Z_y = -(C_{12}C_{13} - C_{11}C_{23}) \tag{4.4}$$

4.2 Simplified Approximate Solution for Isotropic Bimodulus Material

The only bimodulus elliptic plate solution available in the literature is Kamiya's iterative solution [9] for the isotropic circular plate. Thus, as in Section 3.2, it is desirable to derive a simple approximate solution.

Here we consider a uniformly loaded circular plate of isotropic bimodulus material. Due to the axisymmetric nature of the present problem, all derivatives with respect to ϕ vanish and it is more convenient to work with plane polar coordinates (r,ϕ) instead of rectangular (x,y). Thus, the governing equations are

$$\varepsilon_r^0 = u_{r,r} \quad ; \quad \varepsilon_\phi^0 = u_r/r \quad ; \quad \varepsilon_{r\phi}^0 = 0 \tag{4.5}$$

$$\kappa_r = w,_{rr} \quad ; \quad \kappa_\phi = -w,_r/r \quad ; \quad \kappa_{r\phi} = 0 \tag{4.6}$$

$$rN_{r,r} + N_r - N_\phi = 0 \tag{4.7}$$

$$rM_{r,r} + M_r - M_\phi = qr^2/2 \tag{4.8}$$

$$\sigma_r = Q_k(\varepsilon_r + \nu\varepsilon_\phi) \quad ; \quad \sigma_\phi = Q_k(\nu\varepsilon_r + \varepsilon_\phi) \quad (k=c,t) \tag{4.9}$$

Equations (4.5)-(4.9) may be combined and expressed in terms of displacements as follows:

$$\begin{aligned} &A(ru_{r,rr} + u_{r,r} - r^{-1}u_r) - B(rw_{,rrr} + w_{,rr} - r^{-1}w_{,r}) = 0 \\ &B(ru_{r,rr} + u_{r,r} - r^{-1}u_r) - D(rw_{,rrr} + w_{,rr} - r^{-1}w_{,r}) = qr^2/2 \end{aligned} \tag{4.10}$$

Here A, B, and D are as defined in Eqs. (3.17) and ν does not enter into the equations.

The solutions of Eqs. (4.10) are

$$u_r = U[1 - (r/a)^2](r/a) \quad ; \quad w = W[1 - (r/a)^2]^2 \tag{4.11}$$

Substituting Eqs. (4.11) into Eqs. (4.10), one finds that

$$U/W = -4(B/aA) \tag{4.12}$$

$$W = -qa^4/64D^{(R)} \tag{4.13}$$

where $D^{(R)}$ is as defined in Eq. (3.15).

Also

$$\begin{aligned} z_{nr} &= -\varepsilon_r^0/\kappa_r = u_{r,r}/w_{,rr} = -(a/4)(U/W) \\ z_{n\phi} &= -\varepsilon_\phi^0/\kappa_\phi = r^{-1}u_r/r^{-1}w_{,r} = -(a/4)(U/W) \end{aligned} \tag{4.14}$$

Combining Eqs. (4.12) and (4.14), one obtains

$$z_{nr} = z_{n\phi} = B/A = z_n \tag{4.15}$$

Again the final results are exactly analogous to those obtained in Section 3.2 for isotropic bimodulus rectangular plates.

4.3 Numerical Results

Kamiya [9] gave graphical results obtained by an iterative finite-difference method for maximum deflection for three values of E_t/E_c, with ν_t fixed at 1/4. These are compared with the results obtained by exact and approximate closed-form solutions in Table 4.1. Since the principal-stress directions (and principal-strain directions) coincide with the r,ϕ directions, neither shear modulus nor Poisson's ratio enter into the approximate solution, Eq. (4.13), and thus need not be specified. However, both G and ν enter in the exact solution, since it is expressed in rectangular-coordinate displacement components. Furthermore, since the signs of the two principal-strain components are the same, invariance of the expressions for $Q_{112\ell}$ and $Q_{111\ell}$ requires that the appropriate expressions for G_c and G_t in the respective compressive and tensile regions are

$$G_c = E_c/[2(1 + \nu_c)] \quad ; \quad G_t = E_t/[2(1 + \nu_t)] \tag{4.16}$$

These are consistent with Ambartsumyan and Khachatryan's work [24]. As discussed in [24], a different expression is required for the case when the principal-strain components are of opposite sign.

Table 4.1 Dimensionless Maximum Deflections, E_tWh^3/qa^4, for Uniformly Loaded, Clamped Circular Plates of Isotropic Bimodulus Material Having $\nu_t = 0.25$

E_t/E_c	(1)	(2)	(3)
0.5	0.120	0.1171	0.1171
1.0	0.174	0.1758	0.1758
2.0	0.232	0.2636	0.2636

Method:

(1) Taken from Kamiya's graphical results [9].

(2) Exact closed-form solution.

(3) Simplified solution.

It is noted that both of the closed-form solutions, although independently derived and differing in their dependency on shear moduli and Poisson's ratios, give precisely the same numerical results. This verifies the correctness of the expressions derived. The slight difference from Kamiya's numerical results is probably due to a combination of the approximate nature of his finite-difference solution and inability to read his graphical results accurately.

It should also be mentioned that the dimensionless neutral-surface positions obtained by each of the present solutions also exactly concided, giving values of - 0.1126, 0, and 0.09169 for E_t/E_c values of 1/2, 1, and 2, respectively.

Although it was not possible to rigorously derive Eq. (3.14) for the case of an elliptic plate of isotropic bimodulus material, it is expected that the resulting deflection expression would be a first approximation sufficiently accurate for many purposes, i.e.

$$WE_t h^3/qb^4 = \frac{(1/2)(1-\nu^2)}{1 + c^{-4} + (2/3)c^{-2}} \left(D^{(R)}/D_t\right) \qquad (4.17)$$

The results are compared in Table 4.2, where it is seen that a maximum error of only 13% occurs for the ranges of values $1/2 \leq E_t/E_c \leq 2$ and $1/2 \leq c \leq 2$.

Table 4.2. Comparison of Exact and Approximate Expressions for Dimensionless Deflection $WE_t h^3/qb^4$ for Uniformly Loaded, Clamped Elliptic Plates of Isotropic Bimodulus Material Having $\nu_t = 0.25$

Aspect Ratio c	$WE_t h^3/qb^4$ Approximate	Exact
	$E_t/E_c = 0.5$ $(D^{(R)}/D_t = 1.501)$:	
0.5	0.01588	0.01826
1.0	0.1171	0.1171
2.0	0.2540	0.2533
	$E_t/E_c = 2.0$ $(D^{(R)}/D_t = 0.6685)$:	
0.5	0.03574	0.03871
1.0	0.2636	0.2636
2.0	0.5717	0.5707

To illustrate the dramatic difference between actual orthotropic bimodulus composite materials and the isotropic bimodulus single-layer materials discussed above, Table 4.3 gives results for aramid-rubber and polyester-rubber, using elastic properties from Table 3.3. It is again noted that for the single-layer orthotropic case, Z_y must be omitted in the expressions in Appendix B.

It is seen that in the case of both materials, Z_x decreases as the aspect ratio is increased while the dimensionless deflection increases in magnitude as the aspect ratio is increased.

Table 4.3. Fiber-Direction Neutral-Surface Locations and Dimensionless Deflections for Clamped Elliptic Plates of Single-Layer 0° Aramid-Rubber and Polyester-Rubber

Aspect Ratio c	Aramid-Rubber Z_x	Aramid-Rubber $WE_{22t}h^3/qb^4$	Polyester-Rubber Z_x	Polyester-Rubber $WE_{22t}h^3/qb^4$
0.5	0.4446	0.00613	0.3030	0.002520
0.6	0.4444	0.01226	0.3029	0.005161
0.7	0.4441	0.02161	0.3028	0.009392
0.8	0.4438	0.03457	0.3026	0.01564
0.9	0.4434	0.05116	0.3024	0.02428
1.0	0.4430	0.07095	0.3022	0.03555
1.2	0.4420	0.1168	0.3017	0.06614
1.4	0.4408	0.1644	0.3011	0.1056
1.6	0.4395	0.2076	0.3004	0.1497
1.8	0.4381	0.2436	0.2997	0.1934
2.0	0.4365	0.2721	0.2989	0.2330

Results for the case of cross-ply plates with the bottom layer (layer 1) at 0° (aligned with the x axis) and the top layer (layer 2) at 90° are presented in Table 4.4.

It is seen in Table 4.4 that the same trends as those in Table 4.3 are present. Also it is noted that Z_x remains positive and Z_y remains negative throughout the range of parameters covered; thus, the conditions for which the equations given in Appendix B were derived are satisfied.

Table 4.4. Neutral-Surface Locations and Dimensionless Deflections for Clamped Elliptic Plates of Cross-Ply Laminated Aramid-Rubber and of Polyester-Rubber

Aspect Ratio c	Z_x	Z_y	$WE_{22t}h^3/qb^4$
		Aramid-Rubber	
0.5	0.9075×10^{-7}	-0.5609×10^{-7}	0.6045×10^{-3}
0.6	0.3878×10^{-6}	-0.1976×10^{-6}	0.1250×10^{-2}
0.7	0.1319×10^{-5}	-0.5700×10^{-5}	0.2305×10^{-2}
0.8	0.3794×10^{-5}	-0.1426×10^{-5}	0.3909×10^{-2}
0.9	0.9581×10^{-5}	-0.3204×10^{-5}	0.6212×10^{-2}
1.0	0.2181×10^{-4}	-0.6606×10^{-5}	0.9372×10^{-2}
1.2	0.8843×10^{-4}	-0.2297×10^{-4}	0.01888
1.4	0.2787×10^{-3}	-0.6471×10^{-4}	0.03350
1.6	0.7201×10^{-3}	-0.1491×10^{-3}	0.05385
1.8	0.1583×10^{-2}	-0.2968×10^{-3}	0.07980
2.0	0.3062×10^{-2}	-0.5009×10^{-3}	0.1108
		Polyester-Rubber:	
0.5	0.1581×10^{-5}	-0.1288×10^{-5}	0.002557
0.6	0.6588×10^{-5}	-0.5107×10^{-5}	0.005221
0.7	0.2160×10^{-4}	-0.1619×10^{-4}	0.009452
0.8	0.5897×10^{-4}	-0.4317×10^{-4}	0.01561
0.9	0.1389×10^{-3}	-0.9994×10^{-4}	0.02396
1.0	0.2895×10^{-3}	-0.2045×10^{-3}	0.03457
1.2	0.9242×10^{-3}	-0.6339×10^{-3}	0.06171
1.4	0.2120×10^{-2}	-0.1407×10^{-2}	0.09331
1.6	0.3773×10^{-2}	-0.2417×10^{-2}	0.1243
1.8	0.5580×10^{-2}	-0.3455×10^{-2}	0.1508
2.0	0.7263×10^{-2}	-0.4370×10^{-2}	0.1717

5. CONCLUDING REMARKS AND SUGGESTIONS FOR FURTHER RESEARCH

The exact closed-form solutions developed here for small deflection of two classes of bimodulus composite-material plates have been shown to correlate well with existing approximate solutions for isotropic bimodulus plates and with a mixed finite-element solution previously reported in Technical Report No. 3 of this contract [6].

The approximate closed-form solutions derived here from the exact ones for the special case of isotropic bimodulus material have been shown to be sufficiently accurate for many engineering purposes.

The exact closed-form solution presented in Appendix C for an elliptic plate arbitrarily laminated of ordinary (not bimodulus) material, although original, is shown to be inapplicable to bimodulus composite-material plates except those which are single-layer orthotropic or cross-ply laminated.

The following suggestions for further research on bimodulus composite-material plates constitute extensions of the research reported here:

1. Consideration of large deflections, i.e. geometric nonlinearity (a perturbation-technique to accomplish this is under way).
2. Inclusion of thickness-shear deformation (a closed-form solution for the rectangular plate geometry is under way).
3. Addition of thermal-expansion effects due to a thermal gradient through the thickness as well as a mean temperature change.
4. Free-vibration analysis (a closed-form solution looks promising).

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APPENDIX A: NOMENCLATURE

A	stretching stiffness for isotropic bimodulus plate
A_{ij}	stretching stiffness for orthotropic or anisotropic plate
a,b	dimensions of plate in x and y directions (lengths for rectangle; half lengths for ellipse)
B	bending-stretching stiffness for isotropic bimodulus plate
B_{ij}	bending-stretching stiffness for orthotropic or anisotropic plate
C	nonlinearity coefficient in Eq. (3.21)
C_{rs}	coefficients defined in either Eq. (3.6) or Eq. (C-3)
c	a/b
D	bending stiffness for isotropic bimodulus plate
$D^{(R)}$	reduced bending stiffness for isotropic bimodulus plate
D_{ij}	bending stiffness for orthotropic or anisotropic plate
d_x	$\partial(\ )/\partial x$
E_c,E_t	respective compressive and tensile Young's moduli for isotropic bimodulus material
E_{22_c},E_{22_t}	respective compressive and tensile Young's moduli in the direction transverse to the fibers for an orthotropic bimodulus material
G	shear modulus
G_c,G_t	shear moduli for isotropic bimodulus material in the respective compressive and tensile regions
h	total thickness of plate
K	E_cWh^3/qa^4
L_{rs}	linear differential operators
M_i,N_i	stress couples and stress resultants
m,n	integer indices

P	$(q/E_c)(a/h)^4$
Q_c, Q_t	respective compressive and tensile plane-stress-reduced stiffnesses for isotropic bimodulus material
$\bar{Q}, \Delta Q$	$(1/2)(Q_c+Q_t)$, $Q_c - Q_t$
r	radial position
$Q_{ijk\ell}$	plane-stress-reduced stiffnesses for orthotropic or anisotropic material
q	normal pressure
U, V, W	displacement coefficients for rectangular plate
U_i, V_i, W	displacement coefficients for elliptic plate
u, v, w	displacements in x,y,z directions
u_r	radial displacement
x, y, z	plate coordinates in longitudinal, transverse, and downward thickness directions
Z_x, Z_y	z_{nx}/h, z_{ny}/h
z_{nf}	neutral-surface position associated with $\varepsilon_f = 0$
z_{nx}, z_{ny}	neutral-surface positions associated with $\varepsilon_x = 0$ and $\varepsilon_y = 0$
α, β	$m\pi/a$, $n\pi/b$
$\varepsilon_f, \varepsilon_f^0$	fiber-direction strain at arbitrary location and at midplane
$\varepsilon_j, \varepsilon_j^0$	strain component at arbitrary location and at midplane
κ_f	curvature component in fiber direction
κ_j	curvature component
ϕ	angular position
σ_i	stress component
ν	Poisson's ratio
ν_c, ν_t	Poisson's ratios for isotropic bimodulus material in the respective compressive and tensile regions
θ	angle between fiber direction and plate reference direction (x axis)
$(\ )_{,x}$	$\partial(\ )/\partial x$

APPENDIX B: DERIVATION OF THE PLATE STIFFNESSES FOR TWO-LAYER CROSS-PLY LAMINATE OF BIMODULUS MATERIAL

In the solution of problems involving laminates comprised of bimodulus-material layers, it is necessary to evaluate the integral forms involved in the definitions of the plate stiffnesses, Eqs. (2.2). This is accomplished here for the case of a two-layer cross-ply laminate.

Each layer is assumed to be of the same thickness, h/2, and the same orthotropic elastic properties with respect to the fiber direction. Since each layer is oriented at either 0° or 90° to the x axis, the laminate is also orthotropic, i.e. there are no stiffnesses with subscripts 16 and 26.

The bottom layer is denoted as layer 1, i.e. $\ell = 1$ in $Q_{ijk\ell}$, and occupies the thickness space from $z = 0$ to $z = h/2$, where z is measured positive downward from the midplane. The top layer is denoted as layer 2, i.e. $\ell = 2$, and occupies the thickness space from $z = - h/2$ to $z = 0$.

In the general case derived in this derivation, it is assumed that the upper portion of the top layer (ℓ=2) is in compression (k=2 in $Q_{ijk\ell}$) in the fiber direction and that the lower portion of the top layer is in tension (k=1), while the inner portion of the bottom layer (ℓ=1), from $z = 0$ to $z = z_{nx}$, is in compression (k=2), while the outer portion (from z_{nx} to h/2) of layer 1 is in tension (k=1).

Thus, the general integral expression for A_{ij}, the first of Eqs. (2.2), may be taken as the sum of the integrals for each of three regions:

$$A_{ij} = \int_{-h/2}^{h/2} Q_{ijk\ell}\,dz$$

$$= \int_{-h/2}^{z_{ny}} Q_{ij22}\,dz + \int_{z_{ny}}^{0} Q_{ij12}\,dz + \int_{0}^{z_{nx}} Q_{ij21}\,dz + \int_{z_{nx}}^{h/2} Q_{ij11}\,dz \tag{B-1}$$

Since the planar reduced stiffnesses $Q_{ijk\ell}$ are each respectively constant in the appropriate regions, Eq. (B-1) integrates to the following result:

$$A_{ij} = (Q_{ij22} + Q_{ij11})(h/2) + (Q_{ij21} - Q_{ij11})z_{nx} + (Q_{ij22} - Q_{ij12})Z_y \tag{B-2}$$

or, using $Z_x = z_{nx}/h$ and $Z_y = z_{ny}/h$,

$$A_{ij}/h = (1/2)(Q_{ij22} + Q_{ij11}) + (Q_{ij21} - Q_{ij11})Z_x + (Q_{ij22} - Q_{ij12})Z_y \tag{B-3}$$

Similarly

$$B_{ij} = \int_{-h/2}^{h/2} zQ_{ijk\ell}\,dz$$

$$= \int_{-h/2}^{z_{ny}} zQ_{ij22}\,dz + \int_{z_{ny}}^{0} zQ_{ij12}\,dz + \int_{0}^{z_{nx}} zQ_{ij21}\,dz + \int_{z_{nx}}^{h/2} zQ_{ij11}\,dz \tag{B-4}$$

$$= (-\,Q_{ij22} + Q_{ij11})(h^2/8) + (Q_{ij21} - Q_{ij11})(z_{nx}^2/2) + (Q_{ij22} - Q_{ij12})(z_{ny}{}^2/2) \tag{B-5}$$

or

$$B_{ij}/h^2 = (1/8)(-\,Q_{ij22} + Q_{ij11}) + (Q_{ij21} - Q_{ij11})(Z_x^2/2) + (Q_{ij22} - Q_{ij12})(Z_y^2/2) \tag{B-6}$$

Also

$$D_{ij} = \int_{-h/2}^{h/2} z^2 Q_{ijk\ell}\, dz$$

$$= \int_{-h/2}^{z_{ny}} z^2 Q_{ij22}\, dz + \int_{z_{ny}}^{0} z^2 Q_{ij12}\, dz + \int_{0}^{z_{nx}} z^2 Q_{ij21}\, dz + \int_{z_{nx}}^{h/2} z^2 Q_{ij11}\, dz \qquad \text{(B-7)}$$

$$= (Q_{ij22} + Q_{ij11})(h^3/24) + (Q_{ij21} - Q_{ij11})(z_{nx}^{\ 3}/3) + (Q_{ij22} - Q_{ij12})(z_{ny}^{\ 3}/3) \qquad \text{(B-8)}$$

or

$$D_{ij}/h^3 = (1/24)(Q_{ij22} + Q_{ij11}) + (Q_{ij21} - Q_{ij11})(Z_x^3/3) + (Q_{ij22} - Q_{ij12})(Z_y^3/3) \qquad \text{(B-9)}$$

To apply the preceding equations to a single-layer plate with the fibers oriented in the x direction, it is necessary to merely set $Z_y = 0$.

In deriving the preceding equations, it was assumed that $Z_x \geq 0$ and $Z_y \leq 0$. If the final results obtained for Z_x and Z_y were not meeting these conditions, it is clear that the solution obtained would not be valid. Then it would be necessary to investigate one or more of the other three possible cases, namely:

$$Z_x \geq 0 \quad , \quad Z_y \geq 0$$

$$Z_x \leq 0 \quad , \quad Z_y \leq 0$$

$$Z_x \leq 0 \quad , \quad Z_y \geq 0$$

However, in all of the problems described in this paper, the conditions for the case derived in this appendix are satisfied.

APPENDIX C: CLOSED-FORM SOLUTION OF A UNIFORMLY LOADED, CLAMPED ELLIPTIC PLATE ARBITRARILY LAMINATED OF ANISOTROPIC MATERIAL

The solution of Eqs. (2.6) satisfying boundary conditions Eqs. (4.1) can be expressed in the following form:

$$u(x,y) = [(U_1 x/a) + (U_2 y/b)][1 - (x/a)^2 - (y/b)^2]$$

$$v(x,y) = [(V_1 y/b) + (V_2 x/a)][1 - (x/a)^2 - (y/b)^2] \qquad \text{(C-1)}$$

$$w(x,y) = W[1 - (x/a)^2 - (y/b)^2]^2$$

The expression for $w(x,y)$ was originated by G. H. Bryan for isotropic plates, according to Love [21], and was later used by Ohasi [22] and Lekhnitskii [23] for orthotropic and anisotropic plates and by Kicher [16] for antisymmetric cross-ply plates. The portions with coefficients U_1 and V_1 were also used by Kicher. The portions with coefficients U_2 and V_2 are believed to be original in the present work.

Substituting Eqs. (C-1) in Eqs. (2.6), one obtains the following matrix equation

$$[C_{rs}]\ \{U_1, V_1, W, U_2, V_2\}^T = \{0,0,q,0,0\}^T \qquad \text{(C-2)}$$

with coefficients

$C_{11} = 3(A_{11}/a^2) + (A_{66}/b^2)$; $C_{12} = (A_{12} + A_{66})/ab$; $C_{13} = 12(B_{11}/a^3) + 4(B_{12} + 2B_{66})/ab^2$; $C_{14} = 2A_{16}/ab$; $C_{15} = 3(A_{16}/a^2) + (A_{26}/b^2)$;
$C_{21} = C_{12}$; $C_{22} = (A_{66}/a^2) + 3(A_{22}/b^2)$; $C_{23} = 12(B_{22}/b^3) + 4(B_{12} + 2B_{66})/a^2 b$; $C_{24} = (A_{16}/a^2) + 3(A_{26}/b^2)$; $C_{25} = 2(A_{26}/ab)$;

$$C_{31} = 6(B_{11}/a^3) + 2(B_{12} + 2B_{66})/ab^2 \;;\; C_{32} = 6(B_{22}/b^3) + 2(B_{12} + 2B_{66})/a^2b \;;$$

$$C_{33} = 24(D_{11}/a^4) + 24(D_{22}/b^4) + 16(D_{12} + 2D_{66})/a^2b^2 \;;$$

$$C_{34} = 6(B_{16}/a^2b) + 6(B_{26}/b^3) \;;\; C_{35} = 6(B_{16}/a^3) + 6(B_{26}/ab^2) \;;$$

$$C_{41} = C_{14} \;;\; C_{42} = C_{24} \;;\; C_{43} = 2C_{34} \;;\; C_{44} = (A_{11}/a^2) + 3(A_{66}/b^2) \;;$$

$$C_{45} = C_{12} \;;\; C_{51} = C_{15} \;;\; C_{52} = C_{25} \;;\; C_{53} = 2C_{35} \;;$$

$$C_{54} = C_{12} \;;\; C_{55} = 3(A_{66}/a^2) + (A_{22}/b^2) \tag{C-3}$$

To investigate whether or not the solution expressed by Eqs. (C-1) satisfies the homogeneity condition that z_{nf} is independent of x and y, we use

$$\begin{aligned} \varepsilon_f^0 &= \varepsilon_1^0 \cos^2\theta + \varepsilon_2^0 \sin^2\theta - \varepsilon_6^0 \sin\theta \cos\theta \\ \kappa_f &= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta - \kappa_6 \sin\theta \cos\theta \end{aligned} \tag{C-4}$$

where the strain and curvature components are found by substituting Eqs. (C-1) into Eqs. (2.5)

$$\begin{aligned} \varepsilon_1^0 &= (U_1/a)[1 - 3(x/a)^2 - (y/b)^2] - 2(U_2/a)(x/a)(y/b) \\ \varepsilon_2^0 &= (V_1/b)[1 - (x/a)^2 - 3(y/b)^2] - 2(V_2/b)(x/a)(y/b) \\ \varepsilon_6^0 &= - 2(U_1/b)(x/a)(y/b) - 2(V_1/a)(x/a)(y/b) \\ &\quad + (U_2/b)[1 - (x/a)^2 - 3(y/b)^2] + (V_2/a)[1 - 3(x/a)^2 - (y/b)^2] \\ \kappa_1 &= 4(W/a^2)[1 - 3(x/a)^2 - (y/b)^2] \\ \kappa_2 &= 4(W/b^2)[1 - (x/a)^2 - 3(y/b)^2] \\ \kappa_6 &= 8(W/ab)(x/a)(y/b) \end{aligned} \tag{C-5}$$

Substitution of Eqs. (C-5) into Eqs. (C-4) and thence into Eq. (2.9) shows that they only nontrivial conditions for which z_{nf} is independent of x and y is when θ is equal to either 0° or 90° <u>and</u>

$U_2 = V_2 = 0$. In other words, the only bimodulus case to which a form of Eqs. (C-1) is applicable is the cross-ply laminate. This limitation in the bimodulus case does not detract from the use of Eqs. (C-1) for an arbitrarily general laminate in the case of ordinary materials.

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18. Supplementary Notes - Cont'd

Elliptic Plates of Orthotropic Bimodulus Material" by C.W. Bert and S.K. Kincannon in Developments in Mechanics, Vol. 10 (Proceedings, 16th Midwestern Mechanics Conference), Kansas State University, Manhattan, KS, Sept. 19-21, 1979.

The portion of this report dealing with cross-ply elliptic plates is to be presented at the American Society of Civil Engineers Annual Convention and Exposition, Atlanta, GA, Oct. 22-26, 1979 as "Cross-Ply Elliptic Plates of Bimodulus Material" by S.K. Kincannon, C.W. Bert, and V. Sudhakar Reddy.

20. Abstract - Cont'd

rectangular plate subjected to a sinusoidally distributed normal pressure and (2) a fully clamped elliptic plate subjected to uniform normal pressure. For the special case of isotropic bimodulus material, simplified approximate solutions are deduced from the exact ones. Good agreement is obtained among the two solutions presented here, as well as with numerical results existing in the literature for special cases and with the finite-element results. Also, for the first time is presented a closed-form solution for a rectangular plate arbitrarily laminated of anisotropic ordinary (not bimodulus) material.